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AMENABILITY OF *ap* (*S*)

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Abstract
In this paper, we present an important new theorem for amenability of ap(S). The set of all almost periodic functions on S is denoted by ap(S). We develop the fundamental theory of almost peri- odic function on a general semitopological semigroup. Principal result are that ap(S) is amenable if and only if ap(S) = Sap(S) ⊕ F0 [theorem 3.1].
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I. Introduction

In the present paper we shall produce and study amenability of almost periodic semigroup *S*. We recall that $K(S)$ is the intersection of all the ideal of a semigroup S and B(S) is the C ∗-algebra of all bounded complex-valued functions on S. Note that if F is a translation invariant linear subspace of B(S), then Rs and Ls are bounded linear operators on F . For µ ∈ F ∗, the left introversion operator determined by µ is the mapping Tµ : F → B(S)
defined by

$$(T_\mu f)(s) = \mu(L_s f)(f \in F \ , \ s \in S)$$

The right introversion operator determind by µ is the mapping Uµ :
F → B(S) defined by (Uµf)(s) = µ(Rs f).

An m-admissible subalgebra of B(S) is a translation invariant, left m- introverted C ∗-subalgebra of B(S) containing the containing the constant functions [1].

The main pourpose of this paper is to show that if S be a semitopologi- cal semigroup then ap(S) is amenable if and only if ap(S) = Sap(S) ⊕ F0 where Sap(S) is space of strongly almost periodic functions on S. Let F0 = {f ∈ ap(S) , 0 ∈ (Rsf¯)}, where the closure of Rs f is taken in the topology P of pointwise convergence on S [theorem 3.3].

JOHNF. Berglund [1] introduced the concept of weakly almost periodic func- tions in order to show amenability of W ap(S). Hewitt [2] pointed out that the theories of amenability of Sap(S) are the same. We conclude by estab- lishing some properties of ap(S), for example ap(S) is left amenable

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if and only if ap(S) = Sap(S) ⊕ H for some left translation invariant, closed, linear subspace H of ap(S).

II. Definitions and preliminaries

Definition 2.1. Let S be a semitopological semigroup. A function $f \in C(S)$ is said to be almost periodic if the set $R_s f$ of right translates of f is norm relatively compact in $C(S)$.
ap(S) is a translation invariant introverted C ∗-subalgebra of C (S) con- taining the constant functions [1]. Inparticular ap(S) is m-admissible and

$$R_s R_s f \subset R_s f \ , \ R_s L_s f = L_s R_s f$$

$$R_s (f+g) \subset R_s f + R_s g \ , \ R_s(fg) \subset (R_s f)(R_s g)$$

$$R_s(cf) = cR_s f \ , \ R_s \overline{f} = R_s \overline{f}$$

Definition 2.2. A function f ∈ C (S) is said weakly almost periodic if Rsf is weakly (i.e. σ(C (S), C (S)∗)) relatively compact in C (S).

The set of all weakly almost periodic functions on S is denoted by W ap(S). It is clear that ap(S) ⊂ W ap(S), and there are some instances where equality holds. For example, if S is a compact topological semigroup or a totally bounded topological group on the otherhand, if S is a locally compact, non- compact, topological group, then the two space are far from being equal:
in fact, C o(S)\{0} ⊂ W ap(S)\ap(S). Now, we define a type of almost periodicity, called strong almost periodicity, for which the corresponding F -compactification of a semitopological semigroups is a universal topological group compactification of S [3].

Definition 2.3. A finite dimensional unitary representation of a semitopo- logical semigroup S is a homomorphism U from S into the group of unitary operators on some finite dimensional complex Hilbert space H . The map-ping S →< Usx, y >: S → C (x, y ∈ H) are called coefficients of the representation U . If each ceofficient is continuous, then U is said to be continuous [2].

The space Sap(S) of strongly almost periodic functions on S is defined as the closed linear span in C (S) of the set of all coefficients of continuous, finite dimensional, unitary representations of S. Equivalently, Sap(S) is the closed linear span in C (S) of the Matrix entries avising from the contin- uous, finite dimensional, unitary representation of S by [1], the Sap(S) is a translation invariant, introverted C ∗-subalgebra of ap(S) containing the constant function.

Inparticular Sap(S) is m-admissible, and it is easy to see that Sap(S) ⊂ ap(S) ⊂ W ap(S).

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III. Amenability of *ap*(*S*)

Before we state the next theorem we introduce some convenient notation. Let F be an m-admissible subalgebra of C (S) and let μ ∈ LI M (F). Define F0 = {f ∈ F : 0 ∈ (Rs f⁻)}, where the closure of Rs f is taken in the topology P of pointwise convergence on S, and define

$$F_\mu := \{f \in F : \mu(|f|) = 0\} \ , \ \mu \in LI\,M(F)$$

It is readily verified that both F0 and Fμ are left translation invariant, norm closed, conjugate closed subsets of F . Moreover, Fμ is a proper ideal of F , and F0 has all the properties of a proper ideal of F except possibly closure under addition.

The following theorem is stated only for the left amenable case. We leave the formulation and proof the right amenable version to the reader.

Theorem 3.1. Let S be a semitopological semigroup. Then the following statements are equivalent:

i) ap(S) is left amenable;

ii) ap(S) = Sap(S) ⊕ H for some left translation invariant, closed, linear subspace H of ap(S);

iii) there exists a projection of ap(S) onto Sap(S) that commutes with left translation;

iv) if (ψ, X) is an ap(S)-compactification of S, then for any minimal idem- potent e in X the compactification (ρe ◦ ψ, X e) is an Sap-compactification of S, where ρe : X → X e denotes right translation by e. Furthermore, if (i) holds, then for any extreme LI M μ on F

$$F = Sap(S) \oplus F_\mu \ \ and \ \ F_\mu \subset F_0$$

Proof. (i) implies (iv). If ap(S) is left amenable and if e is a minimal idempotent in X , then X e = eX e is a compact topological group and ρe ◦ ψ : S → X e is a continuous homomorphism onto a dense subsemigroup of X e. Therefore (ρe ◦ ψ, X e) is a topological group compactification of S. If (φ, Y) is another such compactification of S, then φ∗(C (Y)) ⊂ Sap(S) ⊂ ap(S), hence there exists a continuous homomorphism π : (ψ, X) → (φ, Y), π(ρe (ψ(s))) = π(ψ(s))π(e) = π(ψ(s)) = φ(s), (s ∈ S). It follows that (ρe ◦ ψ, X e) ≥ (φ, Y). Therefore (ρe ◦ ψ, X e) is a universal topological group compactification of S.

(iv) implies (iii). For any minimal idempotent e in X ; the left intro- version operator Te : ap(S) → ap(S) is a projection since Te2 =Te. Furthermore Te commutes with each Ls. Now, by T ie tz, e extension theorem, ρe ∗(C (X e)) = Re C

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(X). Hence if (iv) holds then Sap(S) = (ρe ◦ ψ)∗(C (X e)) = ψ∗(Re(C (X))). Since ψ∗Re = Teψ, we obtain Sap(S) =Teap(S), which shows that Te is the required projection.

(iii) implies (ii). If P : ap(S) → Sap(S) is a projection onto Sap(S) such that P Ls = Ls P for all s ∈ S, then H = P −1(0) is a closed left translation invariant subspace of Sap such that ap(S) = Sap(S) ⊕ H .

(ii) implies (i). Let P denote the projection of ap(S) onto Sap(S) defined by the direct sum, and let V be the unique invariant mean on Sap(S). For any s ∈ S and f ∈ ap(S), LsP f ∈ Sap(S) and Ls(f − P f) ∈ H , Hence P Lsf − Ls P f = P Ls(f − P f) = 0 therefore P commutes with left trans- lation, hence V ◦ P is a LI M on ap(S). Now assume that (i) through (iv) hold, and let μ be extreme LI M on ap(S). There is a minimal idempotent e ∈ X such that μ(f) = ∫G fˆ(x)dx , (f ∈ ap(S)) where G is the compact topological group X e = eX e, dx is normalized Haar measure on G, and

ψ∗(fˆ) = f for all f ∈ ap(S). Then μ(|f |) = 0 if and only if Re fˆ = 0, and since ψ∗ (Refˆ) = Te f . We see that Fμ = Te −1(0).

By the proof that (iv) implies (iii), ap(S) = Sap(S)⊕Fμ, and since (Rsf −) =

Txf , Fμ ⊂ F0.

Definition 3.2. Let S be a semitopological semigroup and let F be an m-admissible subalgebra of C (S). Any compactification (ψ, X) of S such that ψ∗ (C (X)) = F is called an F -compactification and (ε, SF) is called the canonical F - compactification of S.

The following result is the amenable version of theorem 3.1.

Theorem 3.3. Let S be a semitopological semigroup then ap(S) is amenable if and only if F0 is an ideal in ap(S) and ap(S) = Sap(S) ⊕ F0. In this case F0 = Fμ, where μ is the unique invariant mean on ap(S).

Proof. By theorem 3.1, it sufficies to prove that if ap(S) is amenable then F0 ⊂ Fμ, where μ is the unique invariant mean on ap(S) [4]. Now, if f ∈ F0, then the set I = {x ∈ Sap(S) : Tx f = 0} is nonempty and is a closed left ideal of Sap(S). Hence I contains a minimal idempotent e. Since K (Sap(S)) is a group, it follows form. The last part of the proof theorem 3.1 that Te−1(0) = Fμ . Therefore, f ∈ Fμ , as required.

Example 3.4. Let S = (R+ , +). We shall use theorem 3.3 to show that ap(S) = Sap(R, +)|S ⊕ C o(S). According to theorem 3.3, for any f ∈ ap(S) we may write f = g + h, where g ∈ Sap(S) and Limn→∞ ∥ Rtn h ∥= 0 for some sequence {tn} in

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S. Clearly, h ∈ C o(S) and g may be extended to a function g1 ∈ ap(R, +), g1|R+ = g.

References

[1] J. F. BERGLUND, H. D. JUNGHENN and P. MILNES, Analysis on Semigroup (Wi- ley, 1989).
[2] Hewitt, E. Ross, K. A, Abstract harmonic analysis. Vol. 1, Berlin-Gottingen, Heidel- berg (Springer, 1963).
[3] R. C. Walker, The Stone-Cech Compactification (Springer-Verlag, 1974).
[4] Volker Runde, Lectures on Amenability (Springer-V, 2002).