Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

Minimal Ideals and Minimal Idempotents in Some Compact Semigroups

A. M. Aminpour[1], M. Seilani[2]

[1]*Department of Mathematical sciences and computer, Shahid Chamran University, Ahvaz, Iran*
[2]*Iranian Academic center for education calture and research*

[*]Seilanimaths11@yahoo.com

Abstract

In this paper we present an important new results for study of fuzzy semigroups (both algebric and topological), in particular, fuzzy compact right topological semigroups. Principal result are that every fuzzy compact Hausdorff right topological semigroup $\tilde{S}$ contains at least one minimal idempotent [theorem 4.1] and every fuzzy left (resp. fuzzy right) ideal contains a fuzzy minimal left (resp. right) ideal [corollary 4.2].
Mathematical Society Classification: 2010 MSC.18340.

Key words: Fuzzy minimal ideal, fuzzy minimal idempotent, fuzzy left (right) zero semigroup.

1. Introduction

We shall present our theory in a fairly concrete setting so that our meth- ods and results will be readily accessible. The notion of fuzzyiness was formally introduced by Lotfizadeh in (1965). Fuzzy set theory has been shown tobe a useful tool to describe situation in which the data are imprecise or vague. Azirel Rosenfeld used the ideal of fuzzy set to introduce the notion of fuzzy subgroups. The ideal of fuzzy subsemigroups was also introduced by Kuroki (1981K) uroki characterized several classes of semigroup in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. A semigroup is an algebraic structure consisting of a non-emptyset S together with an associa- tive binary operation. The formal study of semigroups began in the early 20[th] centary. Semigroups play an important rol in mathematical analysis. The know theory for a fuzzy semigroups show that how to fixed the ideal of an extension of a fuzzy ideal in some group (Kuroki, 1981). The main pourpose of this paper is to show that $K(\tilde{S})$, the fuzzy minimal ideal of a fuzzy semigroup $\tilde{S}$ is a group which is a union of all fuzzy minimal left ideal of $\tilde{S}$ whenever $\tilde{S}$ contains a minimal left ideal. Also we fined [theorem 4.1] that a compact Hausdorff right topological semigroup $\tilde{S}$ contains at least one fuzzy minimal idempotent so that every fuzzy left (resp. fuzzy right) ideal of $\tilde{S}$ contains a fuzzy minimal left (resp. fuzzy right) ideal and every fuzzy closed right ideal contains fuzzy minimal closed right ideal [corollary 4.2]. Also we show that a fuzzy compact Hausdorff, semigroup is a group whenever, it is a can- cellative [corollary 4.4].

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org
ISSN1925-931X

2. Definitions and preliminaries

Definition 2.1. Let S be a semigroup and U, V be two fuzzy set in S. We define $UV(z) = \sup_{xy=z}\{\min(U(x), V(y))\}$,

$$Uy(z) = \begin{cases} \sup_{xy=z} U(x) & \text{if there exists such x} \\ 0 & \text{o.w} \end{cases}$$

and

$$xV(z) = \begin{cases} \sup_{xy=z} V(x) & \text{if there exists such x} \\ 0 & \text{o.w} \end{cases}$$

A fuzzy set $\tilde{S} : S \to [0, 1]$ is called a fuzzy subsemigroup of S, if $\tilde{S}(x, y) \geq \min\{\tilde{S}(x), \tilde{S}(y)\}$ for every $x, y \in S$. An element e of $\tilde{S}$ is said to be a fuzzy right (resp. fuzzy left) identity for $\tilde{S}$ if $\tilde{S}(se) = \tilde{S}(s)$ (resp. $\tilde{S}(es) = \tilde{S}(s)$) for all $s \in S$. An element z in $\tilde{S}$ is a fuzzy right zero if $\tilde{S}(sz) = \tilde{S}(z)$ for all $s \in S$. If every member of $\tilde{S}$ is a fuzzy right zero, then $\tilde{S}$ is called a fuzzy right zero semigroup. Fuzzy left zero and fuzzy right zero semigroup are defined analogusly. An element e of $\tilde{S}$ is said to be fuzzy idempotent, if $\tilde{S}(e^2) = \tilde{S}(e)$. The set of all fuzzy idempotent of $\tilde{S}$ denoted by $E(\tilde{S})$. Let $\tilde{T}$ be a nonempty fuzzy subset of $\tilde{S}$. $\tilde{T}$ is said to be

(a) a fuzzy subsemigroup of $\tilde{S}$, if $\tilde{T}$ is a fuzzy semigroup with respect to multiplication in $\tilde{S}$;
(b) a fuzzy left ideal of $\tilde{S}$ if $\tilde{T}(xy) \geq \tilde{T}(y)$;
(c) a fuzzy right ideal of $\tilde{S}$ if $\tilde{T}(xy) \geq \tilde{T}(x)$;
(d) a fuzzy (two-sided) ideal of $\tilde{S}$ if $\tilde{T}(xy) \geq \max\{\tilde{T}(x), \tilde{T}(y)\}$.

$\tilde{S}$ is called fuzzy left (resp. fuzzy right) simple if it has no proper fuzzy left (resp. fuzzy right) ideals. $\tilde{S}$ is fuzzy simple, if it has no proper fuzzy two-sided ideals. A fuzzy left zero semigroup with more than one element is fuzzy left simple, but not fuzzy right simple. If G is group and $\tilde{G}$ be a fuzzy subset of G. Then $\tilde{G}$ is a fuzzy subgroup of G when $\tilde{G}(ab) \geq \min\{\tilde{G}(a), \tilde{G}(b)\}$ for all $a, b \in G$. Obviosly, fuzzy groups are fuzzy left simple, fuzzy right simple, and fuzzy simple. (see theorem 2.3 for more details).
The next proposition provides some easy tests for determing when a given fuzzy semigroup is fuzzy left simple, fuzzy right simple, or fuzzy simple.

Proposition 2.2. *A fuzzy semigroup $\tilde{S}$ is fuzzy left (resp. fuzzy right) simple, if and only if $\tilde{S}t = \tilde{S}$ (resp. $t\tilde{S} = \tilde{S}$) for all $t \in \tilde{S}$. $\tilde{S}$ is fuzzy simple if and only if $\tilde{S}t\tilde{S} = \tilde{S}$ for all $t \in \tilde{S}$.*

Proof. We prove only the fuzzy left simple version. Since $\tilde{S}t$ is a fuzzy left ideal, where $\tilde{S}t(z) = \sup_{xt=z}\tilde{S}(x)$, the necessity is clear. The sufficiency

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

follows from the observation that if $\tilde{L}$ is a fuzzy left ideal and $t \in L$, then $\tilde{S}t \subset \tilde{L}$.

A fuzzy semigroup$\tilde{S}$ is fuzzy right (resp. fuzzy left) cancellative if $r, s, t \in S$ and $\tilde{S}(sr) = \tilde{S}(tr)$ (resp.$\tilde{S}(rs) = \tilde{S}(rt)$) imply $\tilde{S}(s) = \tilde{S}(t)$. A fuzzy semigroup that is both fuzzy left and fuzzy right cancellative, is said to be fuzzy cancellative. A fuzzy left zero semigroup is trivially fuzzy right cancellative. Any fuzzy subsemigroup of a fuzzy group is fuzzy cancellative.

Theorem 2.3. *The following assertions about a fuzzy semigroup $\tilde{S}$ are equivalent.*

i. $\tilde{S}$ *is fuzzy cancellative and fuzzy simple, and contain a fuzzy idem- potent;*
ii. $\tilde{S}$ *is fuzzy left simple and fuzzy right simple;*
iii. $\tilde{S}$ *is fuzzy left simple and contains a fuzzy left identity;*
iv. $\tilde{S}$ *is a fuzzy group.*

Proof. $(i) \Rightarrow (ii)$ Let $e \in E(\tilde{S})$ and $\tilde{S}e(z) = \sup_{xe=z} \tilde{S}(x)$. Since $e^2 = e$, $xe^2 = z$ and $\tilde{S}e^2(z) = \sup_{xe^2=z} \tilde{S}(x)$, hence $\tilde{S}e(z) = (\tilde{S}e \cdot e)(z)$, so that $\tilde{S}e = \tilde{S}$. Similary $\tilde{S} = e\tilde{S}$. Since $\tilde{S}$ is fuzzy simple and hence for all $t \in S$, $\tilde{S}t\tilde{S} = \tilde{S}$ and $\tilde{S}t\tilde{S} = e\tilde{S}$, thus $\tilde{S}t = \tilde{S}$. Now suppose that $\tilde{L}$is fuzzy left deal of $\tilde{S}$, $\tilde{L}\tilde{S} = \tilde{S}$. Therefore $\tilde{S} = \tilde{S}e \subset \tilde{L}$ which shows that $\tilde{S}$ is fuzzy left simple.
$(ii) \Rightarrow (iii)$ Clearly, $\tilde{S}$ is fuzzy left simple, let $s, t \in \tilde{S}$. Since $\tilde{S}t = \tilde{S} = t\tilde{S}$, there exist $r, s \in S$ such that $t = et$, $s = tr$. Then $es = etr = tr = s$ and $\tilde{S}(es) = \tilde{S}(e)$, so e is fuzzy left identity of $\tilde{S}$.
$(iii) \Rightarrow (iv)$ Let e be a fuzzy left identity for $\tilde{S}$ if $s \in \tilde{S}$, then there exist $t, r \in S$ such that $ts = e$ and $rt = e$ or $\tilde{S}(ts) = \tilde{S}(e)$ and $\tilde{S}(rt) = \tilde{S}(e)$. Then $st = est = rtst = ret = rt = e$ and $se = sts = es = s$, hence $\tilde{S}(st) = \tilde{S}(e)$, $\tilde{S}(se) = \tilde{S}(s)$, so e is a fuzzy identity for $\tilde{S}$, so that $\tilde{S}$ is a fuzzy group.
$(iv) \Rightarrow (i)$ obvious.

3. Fuzzy minimal ideals and fuzzy minimal idempotent

In this section we present discussions of fuzzy minimal ideals and fuzzy minimal idempotents and consider existance of minimal idempotents for fuzzy semigroups.

Definition 3.1. A fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of a fuzzy semigroup $\tilde{S}$ is said to be fuzzy minimal if it properly contains no fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of $\tilde{S}$.

Proposition 3.2. *If a fuzzy semigroup $\tilde{S}$ has a fuzzy minimal ideal $\tilde{K}$, then $\tilde{K}$ is the intersection of all fuzzy ideals of $\tilde{S}$. In particular a fuzzy semigroup can have at most one fuzzy minimal ideal.*

Proof. Let $\tilde{I}$ be any ideals of $\tilde{S}$, then $\tilde{I}\tilde{K} \subset \tilde{I} \cap \tilde{K}$ where

$$\tilde{I}\tilde{K}(z) = \sup_{xy=z} (\min\{\tilde{I}(x), \tilde{K}(y)\}) \text{ and } \tilde{I} \cap \tilde{K} = \min_{x \in \tilde{S}} \{\tilde{I}(x), \tilde{K}(x)\}$$

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

so $\tilde{I} \cap \tilde{K}$ is nonempty and hence is an fuzzy ideal since $\tilde{K}$ is fuzzy minimal, $\tilde{K} = \tilde{I} \cap \tilde{K} \subseteq \tilde{I}$.

Notation. The intersection of all the fuzzy ideal of a fuzzy semigroup $\tilde{S}$ is denoted by $K(\tilde{S})$. It follows from proposition 3.2, that either $K(\tilde{S}) = \bar{0}$or $K(\tilde{S})$ is the smallest of $\tilde{S}$ ($\bar{0}$ is zero character function).

Proposition 3.3. *Let $\tilde{S}$ be a fuzzy semigroup. Then the following assertions hold.*
(i) Distinct fuzzy minimal left ideal of $\tilde{S}$ are disjoint;
(ii) A fuzzy left ideal $\tilde{L}$ of $\tilde{S}$ is fuzzy minimal if and only if $\tilde{L}s = \tilde{S}s = \tilde{L}$, for all $s \in \tilde{L}$ Thus fuzzy minimal left ideals are fuzzy left simple;
(iii) For any fuzzy minimal left ideal $\tilde{L}$ of $\tilde{S}$, $\{\tilde{L}s,\ s \in \tilde{S}\}$ is the family of all fuzzy minimal left ideal of $\tilde{S}$;
(iv) If $\tilde{S}$ has a fuzzy minimal left ideal. Then every fuzzy ideal of $\tilde{S}$ contains a fuzzy minimal left ideal;
(v) A fuzzy ideal $\tilde{I}$ of $\tilde{S}$ is fuzzy minimal if and only if $\tilde{S}s\tilde{S} = \tilde{I}$ for all $s \in \tilde{I}$. In this case $\tilde{I}s\tilde{I} = \tilde{I}$ for all $s \in S$. In particular, fuzzy minimal ideals are fuzzy simple.

Proof. (i) Let $\tilde{L}, \tilde{L}'$ are distinct fuzzy minimal left ideal of $\tilde{S}$ such

$$\tilde{L} \cap \tilde{L}' = \min_{x \in \tilde{S}}\{\tilde{L}(x), \tilde{L}'(x)\} = 0.$$

Now $\tilde{L} \cap \tilde{L}$ is a fuzzy left ideal of $\tilde{S}$ such that $(\tilde{L} \cap \tilde{L}')(x) \leq \tilde{L}(x)$ and $(\tilde{L} \cap \tilde{L}')(x) \leq \tilde{L}'(x)$ by minimality $\tilde{L}, \tilde{L}'$, we get $\tilde{L} = \tilde{L} \cap \tilde{L}' = \tilde{L}'$. i.e $\tilde{L} = \tilde{L}'$ which is contradicts to $\tilde{L} = \tilde{L}'$ so $(\tilde{L} \cap \tilde{L}')(x) = 0$.
(ii) If $\tilde{L}$ is a fuzzy minimal left ideal and $s \in \tilde{L}$. Then $\tilde{L}s$ and $\tilde{S}s$ are fuzzy left ideals such that $\tilde{L}s(z) \leq \tilde{L}(z), \tilde{S}s(z) \leq \tilde{L}(z)$ where $\tilde{L}s(z) = \sup_{xs=z} \tilde{L}(x)$ (if there exist such x) and $\tilde{S}s(z) = \sup_{xs=z} \tilde{S}(x)$ then since $\tilde{L}$ is fuzzy minimal $\tilde{L}s = \tilde{L}, \tilde{S}s = \tilde{L}$ i.e $\tilde{L}s = \tilde{S}s = \tilde{L}, \forall s \in \tilde{L}$.
Conversely, if $\tilde{L}s = \tilde{S}s = \tilde{L}$ for every $s \in \tilde{L}$, we show that $\tilde{L}$ is fuzzy minimal. Let $\tilde{J}$ be any fuzzy left ideal and $\tilde{J} \subseteq \tilde{L}$, let $s \in \tilde{J}$ so, $\tilde{L}s = \tilde{L}$ fuzzy left ideal and $\tilde{L}s \subseteq \tilde{J}$ therefor $\tilde{L} \subseteq \tilde{J}$ so $\tilde{J} = \tilde{L}$ i.e $\tilde{L}s$ is fuzzy minimal left ideal of $\tilde{S}$.
Proof (iii), (iv), (v) is clear.

Corollary 3.4. *Let $\tilde{S}$ and $\tilde{T}$ be fuzzy semigroup and θ be a fuzzy homomorphism of $\tilde{S}$ onto $\tilde{T}$. If $\tilde{L}$ is a fuzzy minimal left ideal of $\tilde{S}$, then $\theta(\tilde{L})$ is a fuzzy minimal left ideal of $\tilde{T}$. If $K(\tilde{S}) = \bar{0}$ then $\theta(K(\tilde{S})) = K(\tilde{T})$.*

Proof. $\tilde{L}$ is fuzzy minimal left ideal of $\tilde{S}$ thus $\tilde{L}s = \tilde{S}s = \tilde{L}, \forall s \in \tilde{L}$. We show that $\theta(\tilde{L})$ is fuzzy minimal left ideal of $\tilde{T}$. (i.e $\theta(\tilde{L}) = \tilde{T}t$ for every $t \in \tilde{T}$).

$$\theta(\tilde{L}z) = \theta(\tilde{S}s)(z) = (\theta(\tilde{S})\theta(s))(z) = (T\theta(s))(z)$$

$$= \sup_{x \cdot \theta(s) = z} \tilde{T}(z) = \sup_{x \cdot t = z} \tilde{T}(z) = \tilde{T}t(z)$$

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

To prove $\theta(K(\tilde{S})) = K(\tilde{T})$, we show that $\theta(K(\tilde{S}))$ is minimal ideal of $\tilde{T}$. For this we prove that $\theta(K(\tilde{S})) = \tilde{T}t\tilde{T}$ for every $t \in \tilde{T}$. Since θ is onto, there

exist $s \in \tilde{S}$ such that $K(\tilde{S}) = \tilde{S}s\tilde{S}$ for every $s \in \tilde{S}$. So

$$\theta(K(\tilde{S})) = \theta(\tilde{S}s\tilde{S}) = \theta(\tilde{S})\theta(s)\theta(\tilde{S}) = \tilde{T}t\tilde{T}$$

i.e $\theta(K(\tilde{S}))$ is a fuzzy minimal ideal of $\tilde{T}$. But a fuzzy minimal ideal is unique. So $\theta(K(\tilde{S})) = K(\tilde{T})$.

The next result shows that if a semigroup has a minimal left ideal, then it also has a minimal ideal.

Proposition 3.5. *If a fuzzy semigroup $\tilde{S}$ contains a fuzzy minimal left ideal, then $K(\tilde{S})$ is the union of all fuzzy minimal left ideal of $\tilde{S}$.*

Proof. The proof is straightforward.

The next theorem get us the eqivalent condition for fuzzy minimal idem- potent elements.

Theorem 3.6. *Let e be an fuzzy idempotent in a fuzzy semigroup $\tilde{S}$, then the following assertions are equivalent.*
(i) $\tilde{S}e$ is a fuzzy minimal left ideal;
(ii) $e\tilde{S}$ is a fuzzy minimal right ideal;
(iii) $e\tilde{S}e(= e\tilde{S} \cap \tilde{S}e)$ is a fuzzy group and hence is the maximal fuzzy subgroup of $\tilde{S}$ containing e.

Proof. It is enough to prove that $(i) \Leftrightarrow (iii)$. By symmetry $(ii) \Leftrightarrow (iii)$. Now let (i) holds then $\tilde{S}e$ is a fuzzy minimal left simple i.e $\tilde{S}e(se) = \tilde{S}e$ for every $s \in \tilde{S}$. $(se \in \tilde{S}e)$ and $(\tilde{S}e(se))(z) = \tilde{S}e(z)$. So,

$$\sup_{x \cdot se = z} \tilde{S}e(x) = \sup_{s'e \cdot se = z} \tilde{S}e(s'e).$$

In this case

$$s'e \cdot se = s'e \cdot e \overset{e^2 = e}{\Longrightarrow} es'ese = es'e = (es'e)(ese) = es'e \text{ forall } s' \in S$$

Hence fuzzy subsemigroup $e\tilde{S}e$ is a fuzzy left simple and containing the fuzzy identity, so $e\tilde{S}e$ is a group (by I.10).
$(iii) \Rightarrow (i)$ $\forall s \in \tilde{S}$, $ese \in e\tilde{S}e$. Since eSe is a fuzzy group with fuzzy identity e, so, there exists $r \in S$, such that $(ere)(ese) = e$ (i.e $\tilde{S}((ere)(ese)) = \tilde{S}(e)$) or $\tilde{S}(erese) = \tilde{S}(e)$. It follows that $\tilde{S}(se) = \tilde{S}(sese)$ therefor $\tilde{S}e$ is a fuzzy left simple and hence is a fuzzy minimal left ideal.

Definition 3.7. An idempotent e in a fuzzy semigroup $\tilde{S}$ is said to be minimal if it satisfies the equivalent condition $(i) - (iii)$ of theorem 3.6.

Thus fuzzy minimal idempotents are idempotents that lie in fuzzy minimal left ideals and in fuzzy minimal right ideals.

Theorem 3.8. *The following assertion hold for a fuzzy semigroup $\tilde{S}$ with a fuzzy minimal idempotent.*

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

Journal of Academic and Applied Studies

ISSN1925-931X

(*i*) $\tilde{S}$ *has a unique fuzzy minimal ideal* $K := K(\tilde{S})$;
(*ii*) $E(K) = \bar{0}$, *in fact* $E(K)$ *is the set of fuzzy minimal idempotent of* $\tilde{S}$;
(*iii*) $\{\tilde{S}e : e \in E(K)\}$, $\{e\tilde{S}: e \in E(K)\}$ *and* $\{e\tilde{S}e : e \in E(K)\}$ *are, respectively, the set of fuzzy minimal left ideal of* $\tilde{S}$, *the set of fuzzy minimal right ideals of* $\tilde{S}$ *and the set of maximal subgroup of* K;
(*iv*) $K = \cup\{\tilde{S}e : e \in E(K)\} = \cup\{e\tilde{S} : e \in E(K)\} = \cup\{e\tilde{S}e : e \in E(K)\}$.

Proof. This theorem is an immediate consequence of 3.5 and its dual.

Remark 3.9. *By theorem 1.2.B pointed out by* Berglund (1989) $K(\tilde{S})$ *is a group if and only if* $\tilde{S}$ *has a unique minimal left ideal and a unique minimal right ideal. Thus, if* $\tilde{S}$ *is a fuzzy abelian semigroup and* $K(\tilde{S}) = \bar{0}$ *then* $K(\tilde{S})$ *is a group.*

Corollary 3.10. *Let* $\tilde{S}$ *be a fuzzy semigroup with a fuzzy minimal idempo- tent.*
(*i*) *The fuzzy minimal left ideal of* $\tilde{S}$ *are fuzzy group if and only if* $\tilde{S}$ *has a unique fuzzy minimal right ideal;*
(*ii*) $K(\tilde{S})$ *is a fuzzy group if and only if* $\tilde{S}$ *has a fuzzy unique minimal left ideal and a fuzzy unique minimal right ideal.*

Proof. (*i*) Let $\tilde{S}$ has a unique fuzzy minimal right ideal $\tilde{R}$. Then $\tilde{R} = e\tilde{S}$. So have $K(\tilde{S}) = e\tilde{S} = \tilde{R}$ is the union of fuzzy minimal left ideal $\tilde{S}d$, $d \in E(K)$. But $\tilde{S}d = \tilde{R} \cap \tilde{S}d$. (Note that $\tilde{S}d \subseteq K(\tilde{S}) = \tilde{R}$ i.e $Sd \subseteq \tilde{R} \Rightarrow Sd \in R \cap Se$). $\tilde{R}$ is fuzzy minimal right ideal and $\tilde{S}d$ is a fuzzy minimal left ideal.

$$K(\tilde{S}) = \bigcup_{d \in E(K)} \tilde{S}d = \bigcup_{\tilde{L}\ \text{is fuzzy left minimal ideal}} \tilde{L} = \bigcup_{\tilde{R}\ \text{is fuzzy right minimal ideal}} \tilde{R} = \bigcup e\tilde{S} = \bigcup \tilde{R} = e\tilde{S}$$

There exist d such that $\tilde{S}d \subset \tilde{R}$ then $\tilde{R} \cap \tilde{S}d = \tilde{S}d$ is fuzzy group, and $e\tilde{S}$, $d\tilde{S}$ are two fuzzy minimal right ideal for $\tilde{S}$ ($e, d \in E(K)$). We have $ed \in \tilde{S}d = d\tilde{S}d$ hence $ed = ded$ then $ed\tilde{S} = ded\tilde{S}$. $d\tilde{S} = ded\tilde{S} = ed\tilde{S} = e\tilde{S}$ then $d\tilde{S} = e\tilde{S}$.
(ii) assertion follows immediately from (i) and it dual.

Corollary 3.11. *If* $\tilde{T}$ *be a fuzzy subsemigroup of a fuzzy semigroup* $\tilde{S}$, *and suppose that* $\tilde{S}$ *and* $\tilde{T}$ *have fuzzy minimal idempotents. Then the following assertion hold.*
(*i*) *If* $\tilde{T} \cap K(\tilde{S}) = \bar{0}$, *then* $K(\tilde{T}) = \tilde{T} \cap K(\tilde{S})$;
(*ii*) *If* $\tilde{S}$ *is fuzzy simple, then so is* $\tilde{T}$;
(*iii*) *If* $\tilde{S}$ *is fuzzy left simple, then so is* $\tilde{T}$.

Proof. (*i*) Assume that $\tilde{T} \cap K(\tilde{S}) = 0$. By 3.8 (*iv*), $\tilde{T} \cap K(\tilde{S})$ is the union of sets $\tilde{T} \cap \tilde{S}e$, where $e \in E(K(\tilde{S}))$. Now $\tilde{T} \cap \tilde{S}e$, if nonempty, is a left ideal of $\tilde{T}$ and hence contains a fuzzy minimal left ideal $\tilde{T}d$, where $d \in E(\tilde{T})$ [3.3, (*iv*)]. Since $d \in \tilde{S}e$, $\tilde{S}d = \tilde{S}e$ and hence $\tilde{T}d = \tilde{T} \cap \tilde{S}d = \tilde{T} \cap \tilde{S}e$. Thus $\tilde{T} \cap K(\tilde{S})$ is a union of fuzzy minimal left ideals of $\tilde{T}$, so $\tilde{T} \cap K(\tilde{S}) \subset K(\tilde{T})$. Since $\tilde{T} \cap K(\tilde{S})$ is obviously an ideal of $\tilde{T}$, equality must hold.
(*ii*) If $\tilde{S}$ is fuzzy simple, then $\tilde{T} \cap K(\tilde{S}) = \tilde{T} \cap \tilde{S} = \min\{\tilde{T}(x), \tilde{S}(x)\} = \tilde{T}$, so, by (*i*), $\tilde{T} = K(\tilde{T})$.

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

(iii) If $\tilde{S}$ is fuzzy left simple, then $\tilde{S}(se) = \tilde{S}(s)$ forall $s \in S$ and $e \in E(\tilde{S})$, hence, $\tilde{T}(te) = \tilde{T}(t)$ forall $t \in T$ and $e \in E(\tilde{T})$. Therefor $\tilde{T}$ is fuzzy left simple.

Definition 3.12. A fuzzy left (resp. fuzzy right) group is a fuzzy semi- group $\tilde{S}$ with the property that for each pair of element s, t in $\tilde{S}$ there existsa unique $x \in \tilde{S}$ such that $\tilde{S}(xs) = \tilde{S}(t)$ (resp.$\tilde{S}(sx) = \tilde{S}(t)$). An importantexample of a fuzzy left group is a fuzzy minimal left ideal in a fuzzy semigroup with fuzzy minimal idempotents. The following theorem, which should be compared with theorem 2.3.

Theorem 3.13. *The following assertions about a fuzzy semigroup $\tilde{S}$ are equivalent:*

(i) *$\tilde{S}$ is fuzzy left simple and fuzzy right cancellative;*
(ii) *$\tilde{S}$ is fuzzy left simple and contains an fuzzy idempotent;*
(iii) *$\tilde{S}$ is fuzzy right cancellative and contains a fuzzy minimal idempotent;*
(iv) *$\tilde{S}$ is isomorphic to the direct product of a fuzzy left zero semigroup and a fuzzy group;*
(v) *$\tilde{S}$ is a fuzzy left group.*

Proof. The proof is straightforward.

4. Existence of fuzzy minimal idempotent

General topology is one of the first branches of pure mathematics to which the notion of fuzzy sets has been applied systematically. According to Chang (1968), a fuzzy topological space is a pair (X, F), where X is any set and $F \subset I^X$ ($I = [0, 1]$) satisfying the following axioms:

(i) $\emptyset, X \in F$;
(ii) If $A, B \in F$, then $A \cap B \in F$;
(iii) If $A_i \in F$ for each $i \in I$, then $\vee_i A_i \in F$

Let (X, F_1), (Y, F_2) be two fuzzy topological spaces. A mapping f of (X, F_1) into (Y, F_2) is fuzzy continuous if and only if for each open fuzzy set v in F_2, the inverse image $f^{-1}(v)$ is in F_1. Where $\widehat{f^{-1}(v)}(x) = v(\widehat{f(x)})$ forall $x \in X$. A bijective mapping f is fuzzy homeomorphism if and only if both f and f^{-1} are fuzzy continuous.

By definition a fuzzy topological semigroup $\tilde{S}$ is a Hausdorff space with fuzzy continuous associative multiplication $(x, y) \to xy$ of $\tilde{S} \times \tilde{S}$ into $\tilde{S}$, and if the ultiplication is continuous in each variable separately, $\tilde{S}$ is called a fuzzy semitopological semigroup (Geetha, 1992). recall that if $\tilde{S}$ be a fuzzy semigroup and a fuzzy topological space $\tilde{S}$ is called a fuzzy right topological semigroup if $\rho_s : x \to xs$ from (S, F) to (S, F) is fuzzy continuous for each $s \in \tilde{S}$. Similary $\tilde{S}$ is called a fuzzy left topological semigroup if $\lambda_s : x \to sx$ from (S, F) to (S, F) is fuzzy continuous for each $s \in \tilde{S}$.

The next theorem is a key result in the theory of compact semigroups.

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

Theorem 4.1. *Let $\widetilde{S}$ be a fuzzy compact, Hausdorff, right topological semi- group. Then $\widetilde{S}$ has a fuzzy minimal idempotent, hence $\widetilde{S}$ has a unique fuzzy minimal ideal with structure described in theorem* 4.1. *Moreover, fuzzy min- imal left ideal of $\widetilde{S}$ are closed and pairwise fuzzy homeomorphic and fuzzy subgroup of $K(\widetilde{S})$ lying in the same fuzzy minimal right ideal are fuzzy topo- logically isomorphic.*

Proof. We show first that $\widetilde{S}$ contains an fuzzy idempotent. Order the family $\widetilde{J}$ of fuzzy closed subsemigroup of $\widetilde{S}$ by inclusion (i.e if $J_1, J_2 \in \widetilde{J}$ then $J_1 \subset J_2$ if and only if $J_1(s) \leq J_2(s)$ for every $s \in \widetilde{S}$) if C is a linearly ordered subset of $\widetilde{J}$. Then the compactness of $\widetilde{S}$ implies that $\cap C = 0$, hence C has a lower bound in $\widetilde{J}$. *Zorn's lemma* now guarantees the existence of a fuzzy minimal member $\widetilde{T}$ of $\widetilde{J}$. Let d be any member of $\widetilde{T}$. We shall show that d is an fuzzy idempotent. By continuous $\rho_d : \widetilde{T} \rightarrow \widetilde{T}$ and $\rho_d(\widetilde{T}) = \widetilde{T} \cdot d$, $\widetilde{T} \cdot d$ is a closed subsemigroup of $\widetilde{T}$. Where $(\widetilde{T} \cdot d)(z) = \sup_{x \cdot d = z} \widetilde{T}(x)$ (if there exists such x) and $(\widetilde{T} \cdot d)(z) \leq \widetilde{T}(z)$ for every $z \in \widetilde{S}$. The minimality of the latter implies that $\widetilde{T} \cdot d = \widetilde{T}$. Let $\widetilde{T}_1 := \widetilde{T} \cap \rho^{-1}(d)$. The set $\widetilde{T}_1$ is nonempty.
$\widetilde{T}_1$ is closed fuzzy subsemigroup of $\widetilde{T}$ and $\widetilde{T}_1 = \widetilde{T}$, hence $d \in \widetilde{T}_1$ consider that $\widetilde{T} \circ \rho_d : \widetilde{T} \rightarrow [0, 1]$ and $\widetilde{T}_1(d) = (\widetilde{T} \circ \rho_d)(d) = \widetilde{T}(\rho_d(d)) = \widetilde{T}(d^2)$ there for d is fuzzy idempotent. Now let $\widetilde{J}$ be any fuzzy minimal close left ideal of $\widetilde{S}$. That is a fuzzy closed left ideal that is minimal with respect to these two properties. (The proof of the existence of $\widetilde{J}$ also uses *Zorn's lemma*). let $\widetilde{I}$ be any fuzzy left ideal of $\widetilde{S}$ contained in $\widetilde{J}$. If $s \in \widetilde{I}$ then, $\widetilde{S}s(x) \leq I(x) \leq J(x)$, for every $x \in \widetilde{S}$ and since $\widetilde{S}s$ is a closed fuzzy left ideal of $\widetilde{S}$, $\widetilde{S}s = \widetilde{J}$, hence $\widetilde{I} = \widetilde{J}$. Thus $\widetilde{J}$ is a fuzzy minimal left ideal of $\widetilde{S}$. Since $\widetilde{J}$ is closed the first part of the proof implies that J contains an fuzzy idempotent which is necessarily a fuzzy minimal idempotent of $\widetilde{S}$. This completes the proof of the first part of the theorem. Any fuzzy minimal left ideal $\widetilde{L}$ of $\widetilde{S}$ is of the form $\widetilde{S}e = \rho_e(\widetilde{S})$, where e is a fuzzy minimal idempotent (i.e $\widetilde{S}(e^2) = \widetilde{S}(e)$) hence $\widetilde{L}$ is closed. If $\widetilde{L}$ is another minimal left ideal of $\widetilde{S}$ and $r \in \widetilde{L}'$ then

$$\begin{cases} \widetilde{L} \rightarrow \widetilde{L}' \\ (s, \widetilde{L}(s)) \rightarrow (sr, \widetilde{L}(sr)) \end{cases}$$

is fuzzy homeomorphism of $\widetilde{L}$ onto $\widetilde{L}'$. Thus fuzzy minimal left ideal of $\widetilde{S}$ are pairwise fuzzy homeomorphic. Finally, that maximal subgroup of $K(\widetilde{S})$ lying in the same minimal right ideal are fuzzy topologically isomorphic.

Corollary 4.2. *In a fuzzy compact, Hausdorff, right topological semigroup every fuzzy left (resp. fuzzy right) ideal contains a fuzzy minimal left (resp. right) ideal. Every closed fuzzy right ideal contains a fuzzy minimal closed right ideal.*

Proof. The assertions of the first sentence follow from 4.1 and 3.3 (*iv*) and its dual. The last assertion follows from *Zorn's lemma*.

Corollary 4.3. *A fuzzy compact, Hausdorff, right topological semigroup is left (resp. right) simple if and only if it is fuzzy right (resp. fuzzy left) cancellative.*

Journal of Academic and Applied Studies
(Special Issue on Applied Sciences)
Vol. 3(9) September 2013, pp. 64-72
Available online @ www.academians.org

ISSN1925-931X

Proof. Use 4.1 and 3.12 and its dual.

Corollary 4.4. *A fuzzy compact, Hausdorff, right topological semigroup is a group if and only if it is fuzzy cancellative.*

Proof. Use 4.3 and 2.3.

Corollary 4.5. *Let $\tilde{T}$ be a closed fuzzy subsemigroup of a fuzzy compact, Hausdorff, right topological semigroup $\tilde{S}$. If $\tilde{S}$ is fuzzy left simple, fuzzy right simple, or fuzzy simple, then so is $\tilde{T}$.*

Proof. Use 4.1 and 3.11 and its dual.

References

[1] Berglund, J. F. Jungheen, H. D. and Milnes, P. (1989). Analysis on semigroups, Wiley.
[2] Chang, C. L. (1968). Fuzzy topological space, *J: Math: Anal*, 24, 182–190.
[3] Geetha, S. K. (1992). Studies on fuzzy topological semigroups and related Areas, Cochin India.
[4] Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroup: fuzzy sets and systems, 5, 203–215.
[5] Lotfizade, A. (1965). Fuzzy sets, Information and control.